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THE STABLE PARALLELIZABILITY OF A SMOOTH HOMOTOPY LENS SPACE

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1. Introduction

Which differential manifolds are parallelizable or stably parallelizable? This is one of the classical questions of topology. In the present paper, we shall be concerned with the stable parallelizability of a lens space; that is, the orbit space of a homotopy sphere under a free cyclic group action of prime order. Ewing et al. answered this problem for the classical lens space, that is, the orbit space of the *standard sphere* under a *linear* cyclic group action.

Theorem (Ewing et al. [3]). *The classical (generalized) lens space* $L^{2n-1}(p; b_1, b_2, \ldots, b_n)$ *is stably parallelizable if and only if*

(i) $n \le p$, *and*

(ii) $b_1^{2j} + b_2^{2j} + \cdots + b_n^{2j} = 0 \pmod p$ *for* $j = 1, 2, \ldots, [\frac{1}{2}(n-1)]$. $\square$

A k-dimensional compact oriented differentiable manifold is called *a topological sphere of dimension* k, or simply a k-sphere, if it is homeomorphic to the k-dimensional standard sphere S^k. A k-sphere not diffeomorphic to the standard k-sphere is said to be *exotic*. The first exotic sphere was discovered by Milnor in 1956 [12].

Let $f(z)$ be a polynomial defined by

$$f(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}, \quad n > 2,$$

where $a_1, a_2, \ldots, a_{n+1}$ are integers greater than 1. The associated algebraic variety (= hypersurface) will be written as V_a. Since the origin is the only critical point of $f(z)$, $V_a - \{0\}$ is a smooth manifold of dimension $2n$. The intersection of

V_a with the $(2n+1)$-dimensional standard sphere S^{2n+1},

$$\Sigma_a = \Sigma(a_1, a_2, \ldots, a_{n+1}) = V_a \cap S^{2n+1},$$

called a Brieskorn manifold, has been studied by many people (see [2, 5, 15]). A Brieskorn manifold is called a Brieskorn sphere if it is a homotopy sphere.

Theorem (Brieskorn [2], Hirzebruch and Mayer [6]). *A $(2n-1)$-dimensional topological sphere bounds a parallelizable manifold if and only if it is diffeomorphic to one of the Brieskorn spheres, where $n \neq 2$.* □

A manifold is (stably) parallelizable if and only if its tangent bundle is (stably) trivial.

The construction of the Brieskorn manifold was generalized by Hamm [4] and Randell [15]. Let

$$f_i(z_1, z_2, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{a_{ij}}, \quad i = 1, 2, \ldots, m$$

be polynomials having only one critical point at the origin, where the a_{ij} are integers greater than 1, α_{ij} are real numbers, and $n > 2$. Set

$$V_i = \{z \in \mathbb{C}^{n+m} : f_i(z) = 0\},$$
$$V_a = V_1 \cap V_2 \cap \cdots \cap V_m,$$
$$\Sigma_a = V \cap S^{2(n+m)-1}.$$

Then $V_i - \{0\}$ is a smooth manifold of dimension $2(n+m)-2$. This Σ_a will be called a *generalized Brieskorn manifold*, and a *generalized Brieskorn sphere* if it is a homotopy sphere.

For our purpose, we *assume* that

(I) grad f_1, grad $f_2, \ldots,$ grad f_m are linearly independent at each point in $V_a - \{0\}$, where grad means gradient, and

(II) Σ_a is a homotopy sphere (of dimension $2n-1$).

Remark. Assumption (I) is true if the a_{ij} are independent of i and the real matrix (α_{ij}) has no zero subdeterminant. Of course, it is easy to choose (α_{ij}) with this property. By assumption (I), $V_a = V_1 \cap V_2 \cap \cdots \cap V_m$ is a complete intersection with its only singularity at the origin, so that $\dim \Sigma_a = 2n-1$.

For a given prime p, and a topological sphere Σ_a, define a free Z_p-action on Σ_a as follows: choose natural numbers b_j so that $a_{ij} b_j = h \pmod p$, $h \neq 0 \pmod p$ for all i, j. Let $T(b_1, b_2, \ldots, b_{n+m})$ be a map on $\mathbb{C}^{n+m}$ defined by

$$T(b_1, b_2, \ldots, b_{n+m})(z_1, z_2, \ldots, z_{n+m}) \\ = (\zeta^{b_1} z_1, \zeta^{b_2} z_2, \ldots, \zeta^{b_{n+m}} z_{n+m}),$$

where $\zeta = \exp(2\pi i/p)$. The map $T(b_1, b_2, \ldots, b_{n+m})$ generates a cyclic group Z_p of order p, under which the topological sphere Σ_a is invariant. The orbit space Σ_a/Z_p will be called a *lens space*, and is denoted by $L(p; a; b)$, where b denotes $(b_1, b_2, \ldots, b_{n+m})$, and $T(b_1, b_2, \ldots, b_{n+m})$ will be written as T unless there is an ambiguity.

Note that we can assume $a_{ij} b_j = 1 \pmod p$ for all i, j without any loss of generality by choosing a suitable generator of Z_p.

The lens spaces $L(p; a; b)$ are studied in this paper by using a method initiated in [3], and we will show the following main results:

Theorem 2.6. *A lens space $L(p; a; b)$ is stably parallelizable if and only if*
(i) *$n \leq p$, and*
(ii) *$b_1^{2j} + b_2^{2j} + \cdots + b_{n+m}^{2j} = m \pmod p$ for $j = 1, 2, \ldots, [\frac{1}{2}(n-1)]$.*

Theorem 3.1. *There exists a stably parallelizable $2n-1$-dimensional lens space $L(p; a; b)$ if and only if $n \leq p$.*

Theorem 5.1. *For $p \geq 5$ prime, there is an action $T = T(b_1, b_2, b_3, b_4, b_5)$ on $\mathbb{C}^5$, and there are representatives, Σ_a's, of all 28 diffeomorphism classes of the 7-dimensional homotopy spheres as submanifolds of $\mathbb{C}^5$ such that the restriction of T on each Σ_a is well defined and all induced lens spaces Σ_a/T are stably parallelizable.*

2. An algebraic characterization of stable parallelizability

Let $\Sigma_a = V_a \cap S^{2(n+m)-1}$ be a $(2n-1)$-dimensional generalized Brieskorn sphere and let

$$T(z_1, z_2, \ldots, z_{n+m}) = (\zeta^{b_1} z_1, \zeta^{b_2} z_2, \ldots, \zeta^{b_{n+m}} z_{n+m})$$

be the generator of the cyclic group Z_p which acts freely on Σ_a as before. In [11], the author described the tangent bundle of the lens space $L(p; a; b)$.

Theorem 2.1 (Kwak [11]). *Let $\tau = \tau(L(p; a; b))$, and ε denote the tangent bundle and the trivial one-dimensional real bundle over $L(p; a; b)$ respectively. Then $\tau + \varepsilon + m(\mathrm{re}(\gamma))$ is isomorphic to*

$$\mathrm{re}(\gamma^{b_1} + \gamma^{b_2} + \cdots + \gamma^{b_{n+m}})$$

over $L(p;a;b)$, where γ denotes the canonical complex line bundle over $L(p;a;b)$, and re( ) *denotes the underlying real bundle.* $\square$

Also, by using the $(2n-1)$-universality of the canonical line bundle over a lens space and by computing its characteristic classes, he showed

Theorem 2.2 (Kwak [11]). *The lens space $L(p;a;b)$ is stably parallelizable if and only if $m(\mathrm{re}(\gamma))$ is stably isomorphic to*

$$\mathrm{re}(\gamma^{b_1}+\gamma^{b_2}+\cdots+\gamma^{b_{n+m}})$$

over the standard lens space $L^{2n+1}(p)$, where γ represents the canonical line bundle over $L^{2n-1}(p)$. Furthermore, in this case, we have

$$(1+v^2)^m=(1+b_1^2v^2)(1+b_2^2v^2)\ldots(1+b_{n+m}^2v^2)$$

in $Z_p[v]/(v^n)\approx\bigoplus H^{\mathrm{even}}(L^{2n-1}(p);Z_p)$. $\square$

In [7], Kambe computed $\tilde{K}O(L^{2n-1}(p))$.

Theorem 2.3 (Kambe [7]). *Let p be an odd prime and let $q=\frac{1}{2}(p-1)$, and $n-1=s(p-1)+r$, $0\le r<p-1$. Then*

$$\tilde{K}O(L^{2n-1}(p))=\begin{cases}(Z_{p^{s+1}})^{[r/2]}+(Z_{p^s})^{q-r/2]} & \text{if } n\not\equiv 1 \pmod p\\ Z_2+(Z_{p^{s+1}})^{[r/2]}+(Z_{p^s})^{q-[r/2]} & \text{if } n\equiv 1 \pmod p\,,\end{cases}$$

and the factors $(Z_{p^{s+1}})^{[r/2]}$ and $(Z_{p^s})^{q-[r/2]}$ are generated by $\sigma,\ldots,\sigma^{[r/2]}$ and $\sigma^{[r/2]+1},\ldots,\sigma^q$ respectively, where $\sigma=\mathrm{re}(\gamma)-2$. $\square$

Lemma 2.4. *If the lens space $L(p;a;b)$ is stably parallelizable, then $n\le p$.*

Proof. Let $L(p;a;b)=\Sigma_a/Z_p$ be stably parallelizable, then $m(\mathrm{re}(\gamma))$ is stably isomorphic to

$$\mathrm{re}(\gamma^{b_1})+\mathrm{re}(\gamma^{b_2})+\cdots+\mathrm{re}(\gamma^{b_{n+m}})$$

over the standard lens space $L^{2n-1}(p)$, which gives

$$m(\mathrm{re}(\gamma)-2)=(\mathrm{re}(\gamma^{b_1})-2)+\cdots+(\mathrm{re}(\gamma^{b_{n+m}})-2)$$

in $\tilde{K}O(L^{2n-1}(p))$. Since $\tilde{K}O(L^{2n-1}(p))$ is abelian, we can assume that $b_1\le b_2\le\cdots\le b_{n+m}$. Let Σ' be the space obtained from Σ_a by taking the complex conjugate at ith coordinate, then $(\Sigma_a,\ T(b_1,\ldots,b_i,\ldots,b_{n+m}))$ is equivariantly diffeomorphic to $(\Sigma', T(b_1, \ldots, p-b_i, \ldots, b_{n+m}))$, and their quotient spaces are diffeomorphic. By taking such diffeomorphic copies of Σ_a, if needed, we can assume that $b_1 \le b_2 \le \cdots \le b_{n+m} \le \frac{1}{2}(p-1)$. Since

$$(\mathrm{re}(\gamma^{b_1})-2)+(\mathrm{re}(\gamma^{b_2})-2)+\cdots+(\mathrm{re}(\gamma^{b_{n+m}})-2)$$

can be written as a polynomial in $\sigma = \mathrm{re}(\gamma)-2$ in $\tilde{KO}(L^{2n-1}(p))$, we can set

$$m\sigma = \alpha_{b_{n+m}} + \alpha_{b_{n+m}-1}\sigma + \cdots + \alpha_0 \sigma^{b_{n+m}},$$

so that $\alpha_{b_{n+m}-1} = m \pmod{p^s}$, and all other coefficients are divisible by p^s. For example, α_0 is a multiple of p^s, and α_0 is also the number of b_j's such that $b_j = b_{n+m}$ in $b_1 \le b_2 \le \cdots \le b_{n+m}$, because

$$\mathrm{re}(\gamma^{b_{n+m}})-2 = \sigma^{b_{n+m}} + \text{terms of lower degree of } \sigma.$$

Similarly, for any b with $1 < b \le \frac{1}{2}(p-1)$, the number of copies of $\mathrm{re}(\gamma^b)-2$ in

$$(\mathrm{re}(\gamma^{b_1})-2)+(\mathrm{re}(\gamma^{b_2})-2)+\cdots+(\mathrm{re}(\gamma^{b_{n+m}})-2)$$

must be divisible by p^s. Now let β be the number of copies $\mathrm{re}(\gamma)-2$ in it, then $\beta + \beta' = \alpha_{b_{n+m}-1} = m \pmod{p^s}$, where β' is the coefficient of σ in the polynomial of σ for

$$(\mathrm{re}(\gamma^{b_1})-2)+\cdots+(\mathrm{re}(\gamma^{b_{n+m}})-2)-\beta(\mathrm{re}(\gamma)-2).$$

On the other hand, β' is divisible by p^s, so $\beta = m \pmod{p^s}$. Since the total number of b_i's is $n+m$, $\beta + hp^s = n+m$ for some h, so $n = s(p-1)+r+1 = 0 \pmod{p^s}$. The only possibility is $s=0$, or $s=1$ and $r=0$. In both cases, $n-1$ is less than p. □

The next lemma will be useful to have for the main theorem.

Lemma 2.5 (Ewing et al. [3]). *Let ξ, η be oriented vector bundles over a finite* CW *complex X, and suppose that*

(i) $\dim(X) < 2p+2$, *p an odd prime, and*

(ii) $H^{4*}(X; Z)$ *has no q-torsion for any odd $q<p$.*

If their Pontrjagin classes $P(\xi)$, $P(\eta)$ are equal, then $(\xi-\eta)-(\dim\xi-\dim\eta) \in \tilde{KO}(X)$ is a 2-torsion element. □

Now, we are ready to get the main theorem.

Theorem 2.6. *The lens space $L(p; a; b)$ is stably parallelizable if and only if*

(i) $n \le p$, *and*

(ii) $(1+v^2)^m = (1+b_1^2v^2)(1+b_2^2v^2)\ldots(1+b_{n+m}^2v^2)$ *in* $Z_p[v]/(v^n)$, *or equivalently*,

(ii)′ $b_1^{2j} + b_2^{2j} + \cdots + b_{n+m}^{2j} = m \pmod p$ *for* $j = 1, 2, \ldots, [\frac{1}{2}(n-1)]$.

Proof. The 'only if' part comes from Lemmas 2.2 and 2.4. Let us assume (i) and (ii). Then the mod p Pontrjagin class of $\mathrm{re}(m\gamma)$ is equal to that of $\mathrm{re}(\gamma^{b_1}) + \mathrm{re}(\gamma^{b_2}) + \cdots + \mathrm{re}(\gamma^{b_{n+m}})$. By Lemma 2.5,

$$\mathrm{re}(-(m\gamma) + \gamma^{b_1} + \gamma^{b_2} + \cdots + \gamma^{b_{n+m}}) - 2n$$

is a 2-torsion element in $\widetilde{KO}(L^{2n-1}(p))$. But it is clearly in the image of $\mathrm{re}: \tilde{K}(L^{2n-1}(p)) \to \widetilde{KO}(L^{2n-1}(p))$, which does not contain any 2-torsion element, hence it must be a zero element. Therefore, $m(\mathrm{re}(\gamma))$ is stably isomorphic to $\mathrm{re}(\gamma^{b_1} + \gamma^{b_2} + \cdots + \gamma^{b_{n+m}})$ over $L^{2n-1}(p)$, and $L(p; a; b)$ is stably parallelizable by Lemma 2.2. The equivalence of (ii) and (ii)′ follows from Newton's identity. □

Corollary 2.7. *Let a lens space $L(p; a; b)$ be defined as an orbit space of a Brieskorn sphere. Then, $L(p; a; b)$ is stably parallelizable if and only if*

(i) $n \le p$, *and*

(ii) $b_1^{2j} + b_2^{2j} + \cdots + b_{n+1}^{2j} = 1 \pmod p$ *for* $j = 1, 2, \ldots, [\frac{1}{2}(n-1)]$. □

It is interesting to compare the stable parallelizability of a lens space $L(p; a; b)$ with that of a classical (generalized) lens space $L(p; b_1, b_2, \ldots, b_n)$. Recall that a classical lens space $L(p; b_1, b_2, \ldots, b_n)$ is defined as an orbit space $S^{2n-1}/T(b_1, b_2, \ldots, b_n)$.

Let $L(p; a; b) = \Sigma_a / T(b_1, b_2, \ldots, b_{n+m})$ be a fixed lens space, where $\Sigma_a = V_a \cap S^{2(n+m)-1}$ as before. Then Σ_a is clearly a submanifold of $S^{2(n+m)-1}$ of codimension $2m$, and $S^{2(n+m)-1}$ is invariant under the action $T(b_1, b_2, \ldots, b_{n+m})$. Hence, the lens space $L(p; a; b)$ is actually imbedded in the classical lens space $L^{2(n+m)-1}(p; b_1, b_2, \ldots, b_{n+m})$.

Corollary 2.8. *Let $L(p; a; b)$ be naturally imbedded in a classical lens space $L^{2(n+m)-1}(p; b_1, b_2, \ldots, b_{n+m})$ as above, and let ν be its normal bundle. Then*

(i) *If* $m = 0 \pmod p$, *then $L(p; a; b)$ is stably parallelizable if and only if $L(p; b_1, b_2, \ldots, b_{n+m})$ is stably parallelizable. Furthermore, the normal bundle ν is stably trivial in this case*;

(ii) *If* $m \ne 0 \pmod p$, *then at most one of $L(p; a; b)$ and $L(p; b_1, b_2, \ldots, b_{n+m})$ can be stably parallelizable, and normal bundle ν cannot be stably trivial.* □

3. The existence of a stably parallelizable lens space

In this section, we prove the following existence theorem of stably parallelizable lens spaces:

Theorem 3.1. *There exists a stably parallelizable* $(2n-1)$*-dimensional lens space* $L(p;a;b)$ *if and only if* $n \leq p$.

As a necessary condition for a lens space $L(p;a;b)$ to be stably parallelizable, there must be a solution to the system of equations

$$b_1^{2j} + b_2^{2j} + \cdots + b_{n+m}^{2j} = m \pmod p, \quad j = 1, 2, \ldots, [\tfrac{1}{2}(n-1)],$$

over the field Z_p, and the existence of their common solution comes from the next lemma.

Lemma 3.2. *Let* $p \geq n \geq 3$ *and let* $q = [\frac{1}{2}(n-1)]$. *Then there exists at least one common solution* $(x_1, x_2, \ldots, x_{n+m})$ *with each* $x_i \neq 0$ *in* Z_p *to*

$$\text{(I)} \qquad \begin{cases} x_1^2 + \cdots + x_n^2 + \cdots + x_{n+m}^2 = m \pmod p, \\ x_1^4 + \cdots + x_n^4 + \cdots + x_{n+m}^4 = m \pmod p, \\ \quad \cdots \\ x_1^{2q} + \cdots + x_n^{2q} + \cdots + x_{n+m}^{2q} = m \pmod p, \end{cases}$$

for some $m \geq 1$.

Proof. If $n = p$, then $x_1 = x_2 = \cdots = x_{n+m} = 1$ is a solution for any $m \geq 1$. Let $n < p$, and let $c_1 = 1, c_2, \ldots, c_{(p-1)/2}$ be the quadratic residue in Z_p. Since each equation in (I) is homogeneous of even order, the system (I) can be reduced to the following system:

$$\text{(II)} \qquad \begin{cases} y_1 + \quad y_2 + \cdots + \qquad\quad y_{(p-1)/2} = n + m, \\ y_1 + c_2 y_2 + \cdots + c_{(p-1)/2}\, y_{(p-1)/2} = m \pmod p, \\ y_1 + c_2^2 y_2 + \cdots + c_{(p-1)/2}^2\, y_{(p-1)/2} = m \pmod p, \\ \quad \cdots \\ y_1 + c_2^q y_2 + \cdots + c_{(p-1)/2}^q\, y_{(p-1)/2} = m \pmod p. \end{cases}$$

Indeed, each y_j in the system (II) represents the number of x_i's with $x_i^2 = c_j$ (mod p) in the system (I), so that the first equation in (II) must be added. Now it is enough to show the existence of a solution to the system (II). To do this, it can be considered as a system of equations in the field Z_p, hence it is needed to change the first equation in (II) to the equation in Z_p, i.e.,

$$y_1 + y_2 + \cdots + y_{(p-1)/2} = n + m \pmod p .$$

If the system (II) with this equation instead of its first equation, say (II)', has a solution, then (II) has a solution $(y_1, y_2, \ldots, y_{(p-1)/2})$ with

$$y_1 + y_2 + \cdots + y_{(p-1)/2} = n + m + kp$$

for some k. By taking a sufficient large number y_1, i.e., adding more variables x_j with $x_j^2 = c_1 = 1$ (mod p) in the system (I), we can assume that $k \geq 0$, and then (I) will have a solution with $m + kp$, $k \geq 0$ instead of m. First, consider the case of $q + 1 = \frac{1}{2}(p-1)$, then the coefficient matrix of the system (II)' has a nonzero Vandermonde determinant. So it has a solution. The remaining case of $q + 1 < \frac{1}{2}(p-1)$ can be proved by the same method by taking

$$y_{q+2} = y_{q+3} = \cdots = y_{(p-1)/2} = 0 . \qquad \square$$

Proof of Theorem 3.1. Note that necessity is known, and all one- and three-dimensional lens spaces are parallelizable. For sufficiency, let $3 \leq n \leq p$. By Lemma 3.2, there are $m \geq 1$ and $b_1, b_2, \ldots, b_{n+m}$ nonzeros in Z_p such that

$$b_1^{2j} + b_2^{2j} + \cdots + b_{n+m}^{2j} = m \pmod p , \quad j = 1, 2, \ldots, [\tfrac{1}{2}(n-1)] .$$

We can choose distinct prime numbers $a_1, a_2, \ldots, a_{n+m}$ so that $a_j b_j = 1$ (mod p) for all j. Now, one can construct m polynomials

$$f_i(z_1, z_2, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{a_{ij}} , \quad i = 1, 2, \ldots, m$$

with $a_{ij} = a_j$ for all i and sufficient real numbers α_{ij} so that the matrix (α_{ij}) has no zero subdeterminant. Then, by Hamm's theorem [4, Satz 1.1],

$$\Sigma = V_1 \cap V_2 \cap \cdots \cap V_m \cap S^{2(n+m)-1} ,$$

where $V_i = f_i^{-1}(0)$, is a $(2n-1)$-dimensional topological sphere, and clearly grad f_1, grad $f_2, \ldots,$ grad f_m are linearly independent. Hence, we are done by Theorem 2.6. $\square$

Remark. Theorem 3.1 does *not* hold for the classical lens spaces. For a classical lens space, the corresponding polynomials are

$$b_1^{2j} + b_2^{2j} + \cdots + b_n^{2j} = 0 \pmod p \quad \text{for } j = 1, 2, \ldots, [\tfrac{1}{2}(n-1)]\,.$$

These equations have no common solutions generally. For example, such nonexistence of a common solution is given by the following result [3, Corollary 2.3]: "If $p = 1 \pmod 4$, $\frac{1}{2}(p+1) \le n < p$, and n is odd, then no classical lens space $L(p; b_1, b_2, \ldots, b_n)$ of dimension $2n-1$ can be stably parallelizable."

4. Spaces $L(k; a; b)$ for any integer $k > 1$

Note that one can define a lens space $L(k; a; b)$ as before for *any* positive integer k greater than 1 (not necessarily prime). The tangent bundle τ of such a lens space $L(k; a; b)$ can also be described as follows: $\tau + \varepsilon + m(\mathrm{re}(\gamma))$ is isomorphic to

$$\mathrm{re}(\gamma^{b_1} + \gamma^{b_2} + \cdots + \gamma^{b_{n+m}})\,,$$

where ε is the one-dimensional trivial bundle, and γ is the one-dimensional complex line bundle over $L(k; a; b)$.

Throughout, we write $k = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$; $p_1 < p_2 < \cdots < p_s$, $r_i \ge 1$ in its prime factorization.

First, note that $L(p_i; a; b)$ is a covering space of $L(k; a; b)$.

Lemma 4.1. *Let E, M be smooth manifolds and let $\pi: E \to M$ be a covering projection. If M is stably parallelizable, then so is E.*

Proof. Trivial, because the induced bundle of the tangent bundle of M under the covering projection π is the tangent bundle of E, and the induced bundle of a trivial bundle is trivial. $\square$

Recall that the Lie group Spin(n) is a double covering of the Lie group SO(n). An oriented smooth n-dimensional manifold M is a Spin manifold if the structure group SO(n) of its tangent bundle can be lifted to Spin(n). It is well known that M is a Spin manifold if and only if its second Stiefel–Whitney class $w_2(M) \in H^2(M; Z_2)$ is zero. For example, all stably parallelizable lens spaces are Spin manifolds.

Note that the real projective space $L(2; a; b)$ is parallelizable (or stably parallelizable) only when its dimension is 1, 3, or 7.

Theorem 4.2. *A* $(2n-1)$*-dimensional lens space* $L(k; a; b)$ *is stably parallelizable if and only if*

(i) $b_1^{2j} + b_2^{2j} + \cdots + b_{n+m}^{2j} = m \pmod{k}$, $j = 1, 2, \ldots, [\frac{1}{2}(n-1)]$ *and*

(ii) *if* $p_1 = 2$, *then* $n = 1, 2$, *or* 4 *and* $r_1 = 1$, *if* $p_1 \neq 2$, *then* $n - 1 < p_1$.

Proof. Necessity. Let $L(k; a; b)$ be stably parallelizable. Then (i) follows from the vanishing of its Pontrjagin classes. If $p_1 > 2$, then $L(p_1; a; b)$ is also stably parallelizable, hence $n - 1 < p_1$ by Theorem 2.6. Let $p_1 = 2$. Then $L(2; a; b)$ is stably parallelizable, so that its dimension $2n-1$ is 1, 3, or 7, i.e., $n = 1$, 2, or 4. Now, by Lemma 4.1, it remains to show that a 7-dimensional lens space $L(4; a; b)$ can *not* be stably parallelizable. Suppose that such a lens space $L(4; a; b)$ were stably parallelizable, then, by the same arguments as in the proof of Lemma 2.4, we have

$$m(\mathrm{re}(\gamma) - 2) = (\mathrm{re}(\gamma^{b_1}) - 2) + \cdots + (\mathrm{re}(\gamma^{b_{4+m}}) - 2)$$

in $\tilde{KO}(L^7(4))$, where $1 \leq b_j \leq 2$ for all j. But b_j must be relatively prime to 4, so that all b_j are 1. Hence $4(\mathrm{re}(\gamma) - 2) = 0$ in $\tilde{KO}(L^7(4))$, but $\mathrm{re}(\gamma) - 2$ generates a cyclic subgroup of order 8 of $\tilde{KO}(L^7(4))$ (see [9]). Hence $L(4; a; b)$ is not stably parallelizable.

Sufficiency. It is sufficient to show that $m(\mathrm{re}(\gamma))$ is stably isomorphic to

$$\mathrm{re}(\gamma^{b_1}) + \mathrm{re}(\gamma^{b_2}) + \cdots + \mathrm{re}(\gamma^{b_{n+m}}),$$

over $L^{2n-1}(k)$. Let $p_1 > 2$ and $n - 1 < p_1$. Since $\tilde{K}(L^{2n-1}(k))$ has no 2-torsion for any odd k, this can be done by the same proof of Theorem 2.6. Let $p_1 = 2$. Since every 1- or 3-dimensional lens space is stably parallelizable, it is necessary to consider only the case $n = 4$ with $r_1 = 1$. Let $L = L^7(2k'; a; b)$, where k' is odd, and let $\pi : L^7(2; a; b) \to L$ be the covering projection. Consider two homomorphisms

$$H^2(L^7(2; a; b); Z_2) \underset{\pi^*}{\overset{t^*}{\rightleftarrows}} H^2(L; Z_2),$$

where t is the transfer map. Then $t^*\pi^* = k'$ on $H^2(L; Z_2)$, hence the second Stiefel–Whitney class

$$w_2(L) = k'w_2(L) = t^*\pi^*(w_2(L)) = t^*(w_2(L^7(2; a; b))) = t^*(0) = 0,$$

because $L^7(2; a; b)$ is parallelizable. Therefore L is a Spin mainifold. Letting $\mu : L \to \mathrm{BSpin}$ be a classifying map for the tangent bundle, the first and only obstruction to lifting to $\mathrm{BO}(8, \ldots, \infty)$ is $\mu^*(x)$, where $x \in H^4(\mathrm{BSpin}; Z) \cong Z$ is a generator. By hypothesis (i), the first Pontrjagin class $P_1(L)$ of L is zero. Since the first Pontrjagin class $P_1(\mathrm{BSpin})$ of BSpin generates $H^4(\mathrm{BSpin}; Z_p)$ for all odd primes p, $P_1(\mathrm{BSpin}) = \pm 2^q x$ for some $q \geq 1$. So, $0 = P_1(L) = \mu^*(P_1(\mathrm{BSpin})) =$

$\pm 2^q\mu^*(x)$, so that $\mu^*(x)$ has even order in $H^4(L; Z) \cong Z_{2k'}$. Since $t^*\pi^* = k'$ is not the zero map in $H^4(L; Z) \cong Z_{2k'}$, π^* is not the zero map. Hence, π^* kills only odd torsion. Since $L(2; a; b)$ is parallelizable, this obstruction dies in $L^7(2; a; b)$, i.e., $\pi^*\mu^*(x) = 0$. Hence, $\mu^*(x) = 0$ and L is stably parallelizable. □

5. Examples

Kervaire and Milnor [8] showed that the diffeomorphism classes of k-spheres, $k \neq 3$, form a finite abelian group, denoted θ_k, under the connected sum operation. This group contains the subgroup bP_{k+1} of those k-spheres which bound a parallelizable manifold, and bP_{k+1} is a cyclic subgroup. The orders of these groups are as in Table 1.

5.1. *On the 5-dimensional homotopy sphere*

Ewing et al. [3] showed the non-existence of a standard linear free Z_5-action on the standard sphere S^5 so that its orbit space, i.e., a classical lens space, is stably parallelizable.

Let us consider a Z_5-action $T(b_1, b_2, b_3, b_4)$ on $\mathbb{C}^4$, where the b_i's are either 2 or 3 so that

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = 1 \pmod 5) .$$

Then, for any $a = (a_1, a_2, a_3, a_4)$ of all different prime numbers satisfying $a_1b_1 = a_2b_2 = a_3b_3 = a_4b_4 \pmod 5$ (note that there are infinitely many such (a_i)'s), Σ_a is the standard 5-dimensional sphere (cf. [4, Satz 1.1]) on which the free Z_5-action $T(b_1, b_2, b_3, b_4)$ is non-linear, and the induced lens space $L(5; a; b)$ is stably parallelizable. Furthermore, these lens spaces are not diffeomorphic to any 5-dimensional classical lens space $L(5; q_1, q_2, q_3)$.

5.2. *On the 7-dimensional homotopy spheres*

It is well known (see [5]) that

$$z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_5\bar{z}_5 = 1 ,$$
$$z_1^3 + z_2^{6k-1} + z_3^2 + z_4^2 + z_5^2 = 0$$

Table 1

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 ···
θ_k	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16252	2	16	16 ···
bP_{k+1}	1	1	?	1	1	1	28	1	2	1	992	1	1	1	8128	1	2	1 ···

represents the element $k \cdot g_2$ in bP_8, where g_2 is a generator of the cyclic group bP_8. Let p be a prime greater than 3, and let

$$a = (3, 6k-1, 2, 2, 2), \qquad b = (b_1, b_2, b_3, b_4, b_5).$$

To induce a stably parallelizable lens space, the equations

$$a_i b_i = m \pmod p, \quad 1 \le m \le p-1$$

and

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 1 \pmod p$$

must be satisfied, i.e.,

$$b_1^2 + \cdots + b_5^2 = (m/3)^3 + b_2^2 + 3(m/2)^2 = 1 \pmod p,$$

and

$$(6k-1) \cdot b_2 = m \pmod p, \quad 1 \le m \le p-1.$$

These two equations give

$$(*) \qquad (6k-1)^2 = \frac{36m^2}{36 - 31m^2} \pmod p, \quad 1 \le m \le p-1.$$

Note that $(36m^2, p) = 1$ and $(36 - 31m^2, p) = 1$. To have a solution of the equation $(*)$ for $6k-1$, the Legendre symbol

$$\left(\frac{36(36-31m^2)^{-1}}{p}\right)$$

for the quadratic residue must be 1, by Euler's criterion. Elementary properties of the Legendre symbol give

$$\left(\frac{36(36-31m^2)^{-1}}{p}\right) = \left(\frac{36-31m^2}{p}\right) = (36-31m^2)^{(p-1)/2}.$$

Hence, by Fermat's theorem, the equation has a solution for some m, $1 \le m \le p-1$, if and only if $36 - 31m^2 = k^2 \pmod p$ for some $k \neq 0$ in Z_p. Now, we are interested in the number M of solutions of $36 - 31m^2 = k^2 \pmod p$, $1 \le k$, $m \le p-1$, i.e., solutions of $(k/6)^2 = 1 - 31(m/6)^2 \pmod p$. Set $s = k/6$, $t = m/6$ to get $s^2 + 31t^2 = 1$, and assume that $p \neq 31$. Then

$$\begin{aligned} M &= \sum_{\substack{c+d=1 \\ 1 \le c,d \le p-1}} \#(s^2 = c) \cdot \#(31t^2 = d) \\ &= \sum_{c=2}^{p-1} \#(s^2 = c) \cdot \#(31t^2 = 1 - c) \end{aligned}$$

$$= \sum_{c=2}^{p-1} \left(1 + \left(\frac{c}{p}\right)\right) \cdot \left(1 + \left(\frac{31^{-1}(1-c)}{p}\right)\right)$$
$$= \sum_{c=2}^{p-1} \left\{1 + \left(\frac{c}{p}\right) + \left(\frac{31^{-1}(1-c)}{p}\right) + \left(\frac{c}{p}\right) \cdot \left(\frac{31^{-1}(1-c)}{p}\right)\right\}$$
$$= (p-2) + \sum_{c=2}^{p-1} \left(\frac{c}{p}\right) + \left(\frac{31^{-1}}{p}\right) \sum_{c=2}^{p-1} \left(\frac{1-c}{p}\right)$$
$$+ \left(\frac{31^{-1}}{p}\right) \sum_{c=2}^{p-1} \left(\frac{c-c^2}{p}\right),$$

where $\#(s^2 = c)$ is the number of solutions of $s^2 = c$, the same for $\#(31t^2 = d)$, and note that the Legendre symbol

$$\left(\frac{c}{p}\right) = \begin{cases} 1 & \text{if } s^2 = c \text{ has (exactly two) solutions,} \\ -1 & \text{otherwise,} \end{cases}$$

so that $\#(s^2 = c) = 1 + (\frac{c}{p})$. But

$$\sum_{c=1}^{p-1} \left(\frac{c}{p}\right) = \sum_{c=2}^{p} \left(\frac{1-c}{p}\right) = 0\,,$$
$$\left(\frac{31^{-1}}{p}\right) = \left(\frac{31}{p}\right), \qquad \left(\frac{1}{p}\right) = \left(\frac{p-1}{p}\right) = 1\,,$$

and

$$\sum_{c=2}^{p-1} \left(\frac{c-c^2}{p}\right) = \sum_{c=1}^{p=1} \left(\frac{c-c^2}{p}\right) = -(-1)^{(p-1)/2}\,,$$

by Jacobstal's theorem; hence we get

$$M = \begin{cases} p - 3 - 2\left(\frac{31}{p}\right) & \text{if } \frac{1}{2}(p-1) \text{ is even}\,, \\ p - 3 & \text{if } \frac{1}{2}(p-1) \text{ is odd, } p \neq 31\,. \end{cases}$$

If $p = 31$, then $(6k-1)^2 = 36m^2/(36 - 31m^2) = m^2 \pmod{31}$. Hence, k can be any number in Z_{31} except $6^{-1} = 26$. Since $M \geq 1$ for $p \geq 7$, there are $6k-1$, m, and $b = (b_1, b_2, \ldots, b_5)$ such that the induced lens space $L(p; 3, 6k-1, 2, 2, 2; b_1, b_2, b_3, b_4, b_5)$ is stably parallelizable. Furthermore, if $p > 7$, there is a solution of

$$x = k \pmod{28}\,,$$
$$x = 6^{-1}(b_1^{-1}m + 1) \pmod{p}$$

by the Chinese Remainder Theorem.

Theorem 5.1. *For $p \geq 5$ prime, there is an action $T = T(b_1, b_2, b_3, b_4, b_5)$ on $\mathbb{C}^5$, and there are representatives, Σ_a's, of all 28 diffeomorphism classes of the*

7-dimensional homotopy spheres as submanifolds of $\mathbb{C}^5$ *such that the restriction of* T *on each* Σ_a *is well defined and all induced lens spaces* Σ_a/T *are stably parallelizable.*

Proof. We have proved this already for $p > 7$. Let $p = 5$, and take $b = (2, 2, 1, 1, 1)$ as a solution of

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 1 \pmod 5 .$$

We choose $a(k) = (21 + 22 \cdot 5k,\ 11,\ 2,\ 2,\ 2)$ so that $a_j b_j = 2 \pmod 5$ for all j, and then $\Sigma_{a(k)} = (15 + 19k) \cdot g_2$ in bP_8 (see [6, Section 14]). Since $(19, 28) = 1$, the $\Sigma_{a(k)}$'s, $k = 1, 2, \ldots, 28$ represent the 28 classes of the 7-dimensional spheres, and all their orbit spaces $L(5; 21 + 110k,\ 11,\ 2,\ 2,\ 2;\ 2,\ 2,\ 1,\ 1,\ 1)$ are stably parallelizable. Finally, for $p = 7$, fix an action $T(b)$ on $\mathbb{C}^5$ with $b = (1, 1, 3, 3, 3)$, which is a solution of

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 1 \pmod 7 .$$

Note that (cf. [6, Section 14])

$$\Sigma_{a(k)},\ a(k) = (13,\ 69 + 182k, 2, 2, 2) \quad \text{represents} \quad 7kg_2 ,$$

$$\Sigma_{a(k)},\ a(k) = (13, 279 + 182k, 2, 2, 2) \quad \text{represents} \quad (1 + 7k)g_2 ,$$

$$\Sigma_{a(k)},\ a(k) = (13, 419 + 182k, 2, 2, 2) \quad \text{represents} \quad (2 + 7k)g_2 ,$$

$$\Sigma_{a(k)},\ a(k) = (13, 489 + 182k, 2, 2, 2) \quad \text{represents} \quad (3 + 7k)g_2 ,$$

$$\Sigma_{a(k)},\ a(k) = (13, 629 + 182k, 2, 2, 2) \quad \text{represents} \quad (4 + 7k)g_2 ,$$

$$\Sigma_{a(k)},\ a(k) = (13,\ 41 + 182k, 2, 2, 2) \quad \text{represents} \quad (5 + 7k)g_2 ,$$

and

$$\Sigma_{a(k)},\ a(k) = (13, 111 + 182k, 2, 2, 2) \quad \text{represents} \quad (6 + 7k)g_2$$

in bP_8. On all the listed spheres, the action $T(b)$ is well defined and all their orbit spaces are stably parallelizable. □

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